

L'Hospital Rule

A special kind of rule that allows evaluating limits of **indeterminate** forms.

If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

Basic Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$

Other Indeterminate forms: $0 \cdot \infty$, 1^∞ , 0^0 , $\infty - \infty$, ∞^0 .

Examples for Basic Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$

It looks like fractions, but we don't use Quotient Rule

① Find $\lim_{x \rightarrow 1} \frac{\ln x}{2x-2}$.

Note: $\frac{\ln 1}{2-2} = \frac{0}{0} \rightsquigarrow$ indeterminate form.

So, we apply L'Hospital Rule:

$$\lim_{x \rightarrow 1} \frac{\ln x}{2x-2} = \lim_{x \rightarrow 1} \frac{1/x}{2} = \lim_{x \rightarrow 1} \frac{1}{2x} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

② Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 4x}$.

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 4x} \quad \left[\frac{0}{0} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\cos 3x \cdot 3}{\sec^2 4x \cdot 4} = \frac{1 \cdot 3}{1 \cdot 4} = \frac{3}{4}$$

③ Find $\lim_{x \rightarrow \infty} \frac{e^x}{x}$.

$$\lim_{x \rightarrow \infty} \frac{e^x}{x} \quad \left[\frac{\infty}{\infty} \right]$$
$$= \lim_{x \rightarrow \infty} \frac{e^x}{1} = \frac{\infty}{1} = \infty$$

④ Find $\lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{\sqrt{x}}$.

$$\lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{\sqrt{x}} \quad \left[\frac{\infty}{\infty} \right]$$
$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{2}(x)^{-1/2}}$$
$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x \ln x}}{\frac{1}{2\sqrt{x}}}$$
$$= \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x \ln x}$$
$$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x} \ln x} = \frac{2}{\infty} = 0$$

Note :- For any other form we need to bring those indeterminate forms to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ & then apply L'Hospital.

⊛ Don't apply L'Hospital directly on any indeterminate form.

Form $0 \cdot \infty$:

$$\text{find } \lim_{x \rightarrow \infty} x e^{-x^2}.$$

Note: direct substitution gives $\infty \cdot 0$

$$\lim_{x \rightarrow \infty} x \cdot e^{-x^2} \quad [\infty \cdot 0]$$

$$= \lim_{x \rightarrow \infty} x \cdot \frac{1}{e^{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} \quad \left[\frac{\infty}{\infty} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{1}{e^{x^2} \cdot 2x}$$

$$= \frac{1}{\infty \cdot \infty} = 0.$$

Form $\infty - \infty$:

$$\text{find } \lim_{x \rightarrow \infty} (x - \ln x).$$

$$\lim_{x \rightarrow \infty} (x - \ln x) \quad [\infty - \infty]$$

$$= \lim_{x \rightarrow \infty} x \left(1 - \frac{\ln x}{x} \right)$$

$$= \infty \cdot (1 - 0)$$

$$= \infty$$

$$\text{Since } \lim_{x \rightarrow \infty} \frac{\ln x}{x} \quad \left[\frac{\infty}{\infty} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{1/x}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Form 1^∞ , ∞^0 , 0^0 ;

Note: The procedure is same for all these types.

Find $\lim_{x \rightarrow \infty} x^{1/x}$.

$$\lim_{x \rightarrow \infty} x^{1/x} \quad [\infty^0]$$

First we consider $y = x^{1/x}$, then take \ln on both sides

ie. $\ln y = \ln(x^{1/x}) = \frac{1}{x} \ln x$ [by properties of \ln]

Now we take limit on both side:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\ln y) &= \lim_{x \rightarrow \infty} \left(\frac{1}{x} \ln x \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{\ln x}{x} \right) \quad \left[\frac{\infty}{\infty} \right] \\ &= \lim_{x \rightarrow \infty} \left(\frac{1/x}{1} \right) = 0 \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\ln y) = 0$$

Since \ln is continuous, then $\lim_{x \rightarrow a}$ & \ln commutes.

ie. $\lim_{x \rightarrow a} (\ln f(x)) = \ln(\lim_{x \rightarrow a} f(x))$

$$\Rightarrow \ln(\lim_{x \rightarrow \infty} y) = 0$$

$$\Rightarrow \lim_{x \rightarrow \infty} y = 1$$

ie. $\lim_{x \rightarrow \infty} x^{1/x} = 1$